

New Gauge Anomalies
and
Topological Invariants in Various Dimensions

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Abstract

In the model of extended non-Abelian tensor gauge fields we have found new metric-independent densities: the exact $(2n + 3)$ -forms and their secondary characteristics, the $(2n + 2)$ -forms as well as the exact $6n$ -forms and the corresponding secondary $(6n - 1)$ -forms. These forms are the analogs of the Pontryagin densities: the exact $2n$ -forms and Chern-Simons secondary characteristics, the $(2n - 1)$ -forms. The $(2n + 3)$ - and $6n$ -forms are gauge invariant densities, while the $(2n + 2)$ - and $(6n - 1)$ -forms transform non-trivially under gauge transformations, that we compare with the corresponding transformations of the Chern-Simons secondary characteristics. This construction allows to identify new potential gauge anomalies in various dimensions.

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1 Introduction

It is well known that one can determine all chiral anomalies, Abelian and non-Abelian [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17], by a differential geometric method without having to evaluate the Feynman diagrams. The non-Abelian anomaly in $2n$ -dimensional space-time may be obtained from the Abelian anomaly in $2n + 2$ dimensions by a series of reduction (transgression) steps [6, 7, 8, 9, 11, 12, 13, 14, 17]. The reduction allows to construct topological densities which are the non-Abelian anomalies and can be represented in a compact integral form [6, 7, 8, 9, 11, 12, 13, 14, 17]. The topological character of these densities has physical relevance [18, 19, 20, 21] and imposes consistency restrictions on the quantum gauge field theories [22, 23, 24]. It also provides various topological mass-generation mechanisms in gauge theories [25, 26, 27]. For instance, in the topologically massive gauge theory in three dimensions, a Chern-Simons term included in the action makes gauge fields massive [25, 26, 27]. Furthermore, in a four-dimensional Abelian gauge field theory, a topological entity called BF term plays the role of a Chern-Simons term and generates a massive vector field [28, 29, 30, 31, 32, 34, 35, 36]. Generalization to the non-Abelian case was recently suggested in [37].

Our intention in this article is to extend these constructions to the non-Abelian tensor gauge fields. Indeed, we found two series of invariant densities in various dimensions which are analogous to the Pontryagin-Chern-Simons densities. First we shall review the lower-dimensional case and then turn to the higher-dimensional extensions.

In the non-Abelian tensor gauge theory [38, 39, 40] there exists a gauge invariant metric-independent density $\Gamma(A)$ in five-dimensional space-time² [37]:

$$\Gamma(A) = \varepsilon^{lmnpq} \text{Tr}(G_{lm}G_{np,q}) = \partial_l \Sigma^l, \quad (1.1)$$

which is the derivative of the vector current Σ_l :

$$\Sigma^l = \varepsilon^{lmnpq} \text{Tr}(G_{mn}A_{pq}). \quad (1.2)$$

The current Σ^l is linear in the Yang-Mills (YM) field-strength tensor G_{mn} and in the rank-2 gauge field A_{pq} which has a symmetric and antisymmetric part, and only its antisymmetric part involved in (1.2). $G_{np,q}$ is the field-strength tensor of the rank-2 gauge field (2.1). The density $\Gamma(A)$ is diffeomorphism-invariant and does not involve the space-time metric. It is also invariant under the group of gauge transformations (2.4): $\Gamma(A^U) = \Gamma(A)$. It

²The definition of the higher-rank field-strength tensors $G_{nm,q}$ is given in eq. (2.1) and we use Latin letters to numerate five-dimensional coordinates x_l ($l, m, n, \dots = 0, 1, \dots, 4$).

shares therefore many properties of the Chern-Pontryagin density in four-dimensional YM theory [18, 13]:

$$\mathcal{P}(A) = \frac{1}{4} \varepsilon^{\mu\nu\lambda\rho} \text{Tr} G_{\mu\nu} G_{\lambda\rho} = \partial_\mu C^\mu, \quad (1.3)$$

which is a derivative of the Chern-Simons topological vector current [18, 13, 19, 20, 41, 42, 43]

$$C^\mu = \varepsilon^{\mu\nu\lambda\rho} \text{Tr} (A_\nu \partial_\lambda A_\rho - i \frac{2}{3} g A_\nu A_\lambda A_\rho). \quad (1.4)$$

Indeed, comparing the expressions (1.1), (1.3) and (1.2), (1.4) one can see that both entities $\mathcal{P}(A)$ and $\Gamma(A)$ are metric-independent, insensitive to the local variation of the fields and are derivatives of the corresponding vector currents C^μ and Σ^l . *The difference between them is that the former is defined in four dimensions, while the latter in five.* This difference in one unit of the space-time dimension originates from the fact that we have at our disposal high-rank tensor gauge fields to build new invariants [37].

While the invariant $\Gamma(A)$ and the vector current Σ^l are defined on a five-dimensional manifold, one can restrict the latter to a lower, four-dimensional manifold. The restriction proceeds as follows. Considering the fifth component of the vector current Σ^l

$$\varepsilon^{4nmpq} \text{Tr} (G_{nm} A_{pq}) \quad (1.5)$$

one can see that the remaining indices will not repeat the external index and the sum is restricted to indices of four-dimensional space-time. Therefore, we can reduce this functional to four dimensions, considering gauge fields independent on the fifth coordinate x_4 . This density is well defined in four-dimensions and is gauge invariant under infinitesimal gauge transformations up to a total divergence term, as one can see below from (1.8). Therefore we shall consider its integral over four-dimensional space-time³ [37]:

$$\Sigma(A) = \frac{1}{32\pi^2} \int_{M_4} d^4x \varepsilon^{\nu\lambda\rho\sigma} \text{Tr} (G_{\nu\lambda} A_{\rho\sigma}). \quad (1.6)$$

This entity is an analog of the Chern-Simons integral⁴

$$W(A) = \frac{g^2}{8\pi^2} \int_{M_3} d^3x \varepsilon^{ijk} \text{Tr} (A_i \partial_j A_k - i g \frac{2}{3} A_i A_j A_k), \quad (1.7)$$

but, *importantly, instead of being defined in three dimensions it is defined in four dimensions.* Thus, the non-Abelian tensor gauge fields allow to build a natural generalization of the Chern-Simons characteristic in four-dimensional space-time.

³We are using Greek letters to numerate the four-dimensional coordinates x_μ ($\mu, \nu, \lambda, \dots = 0, 1, 2, 3$).

⁴The C^3 component of the topological current (1.4) [42, 43, 11].

The functional $\Sigma(A)$ is invariant under infinitesimal gauge transformations up to a total divergence term. Indeed, its gauge variation under δ_ξ , defined in (A.1)-(A.2), is

$$\begin{aligned}\delta_\xi \Sigma(A) &\propto \varepsilon^{\nu\lambda\rho\sigma} \int_{M_4} Tr(-ig[G_{\nu\lambda} \xi]A_{\rho\sigma} + G_{\nu\lambda}(\nabla_\rho \xi_\sigma - ig[A_{\rho\sigma} \xi]))d^4x = \\ &= \varepsilon^{\nu\lambda\rho\sigma} \int_{M_4} \partial_\rho Tr(G_{\nu\lambda}\xi_\sigma)d^4x = \varepsilon^{\nu\lambda\rho\sigma} \int_{\partial M_4} Tr(G_{\nu\lambda}\xi_\sigma)d\sigma_\rho = 0.\end{aligned}\quad (1.8)$$

Here, the first and the third terms cancel each other and the second one, after integration by part and recalling the Bianchi identity (A.3), leaves only a boundary term which vanishes when the gauge parameter $\xi_\sigma(x)$ tends to zero sufficiently fast at the boundary. Hence, the functional is invariant against *small* gauge transformations, but not under *large* ones for which gauge transformations have a non-trivial behavior at the boundary. Thus, we have to find out how $\Sigma(A)$ transforms under large gauge transformations. The expression we found has the form (3.6):

$$\Sigma(A^U) - \Sigma(A) = \frac{i}{32\pi^2 g} \int_{M_4} d^4x \varepsilon^{\mu\nu\lambda\rho} \partial_\lambda Tr(G_{\mu\nu} U_\rho U^-). \quad (1.9)$$

It reduces to (1.8) for the infinitesimal gauge transformations (2.5) and allows to introduce a lower-dimensional density

$$\sigma_3^1(A, U) = \varepsilon^{ijk} Tr(G_{ij} U_k U^-). \quad (1.10)$$

The expression (1.9) is analogous to the corresponding one of the Chern-Simons integral [18, 13, 19, 20, 41, 42, 11, 43]:

$$\begin{aligned}W(A^U) - W(A) &= \frac{1}{8\pi^2} \int_{M_3} d^3x \varepsilon^{ijk} \partial_i Tr(\partial_j U U^- A_k) \\ &+ \frac{1}{24\pi^2} \int_{M_3} d^3x \varepsilon^{ijk} Tr(U^- \partial_i U U^- \partial_j U U^- \partial_k U),\end{aligned}\quad (1.11)$$

and the density (1.10) to the non-Abelian anomaly in two-dimensions [6, 7, 8, 9, 11]:

$$\omega_2^1(A, U) = \varepsilon^{jk} Tr(\partial_j U U^- A_k).$$

Indeed, the above consideration has deep relation with chiral anomalies appearing in gauge theories interacting with Weyl fermions. The Abelian $U_A(1)$ anomaly appears in the divergence of the axial $U(1)$ current $J_\mu^A = \bar{\psi} \gamma_\mu \gamma_5 \psi$, and in four-dimensions it is given by the divergence

$$\partial^\mu J_\mu^A = -\frac{1}{16\pi^2} \varepsilon^{\mu\nu\lambda\rho} Tr(G_{\mu\nu} G_{\lambda\rho}) = -\frac{1}{4\pi^2} \varepsilon^{\mu\nu\lambda\rho} \partial_\mu Tr(A_\nu \partial_\lambda A_\rho - i\frac{2}{3} g A_\nu A_\lambda A_\rho). \quad (1.12)$$

Similarly, the non-Abelian anomaly appears in the covariant divergence of the non-Abelian left $J_\mu^{aL} = \bar{\psi}_L \gamma_\mu \gamma_5 \sigma^a \psi_L$ or right $J_\mu^{aR} = \bar{\psi}_R \gamma_\mu \gamma_5 \sigma^a \psi_R$ handed currents, such as

$$D^\mu J_\mu^{aL} = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} \partial_\mu \text{Tr}[\sigma^a (A_\nu \partial_\lambda A_\rho - i\frac{1}{2} g A_\nu A_\lambda A_\rho)]. \quad (1.13)$$

The Abelian anomaly is gauge invariant, while the non-Abelian anomaly is gauge covariant and is given by the covariant divergence of a non-Abelian current. These lower-dimensional densities have their higher-dimensional counterparts [6, 7, 8, 9, 11, 13, 14, 17]. In $\mathcal{D} = 2n$ dimensions, the $U_A(1)$ anomaly is given by a $2n$ -form, the higher-dimensional analog of eq. (1.12):

$$d * J^A \propto \text{Tr}(G^n) = d \omega_{2n-1}, \quad (1.14)$$

where ω_{2n-1} is a generalization of the Chern-Simons form to $2n - 1$ dimensions [6, 13]:

$$\omega_{2n-1}(A) = n \int_0^1 dt \text{Tr}(A G_t^{n-1}). \quad (1.15)$$

Here, we are using a shorthand notation for the 2-form YM field-strength tensor $G = dA + A^2$ of the 1-form vector field $A = -ig A_\mu^a L_a dx^\mu$, with $G_t = tG + (t^2 - t)A^2$.

Our aim is to generalize the above construction (1.1), (1.6), (1.9) and (1.10) by defining invariant densities in higher dimensions $\mathcal{D} = 2n + 3 = 5, 7, 9, 11, \dots$:

$$\Gamma_{2n+3}(A) = \text{Tr}(G^m G_3) = d \sigma_{2n+2}, \quad (1.16)$$

where we are using a shorthand notation for the 3-form field-strength tensor $G_3 = dA_2 + [A, A_2]$ of the rank-2 gauge field $A_2 = -ig A_{\mu\nu}^a L_a dx^\mu \wedge dx^\nu$ and $G_{3t} = tG_3 + (t^2 - t)[A, A_2]$. The $(2n + 2)$ -form σ_{2n+2} is:

$$\sigma_{2n+2}(A, A_2) = \int_0^1 dt \text{Tr}(A G_t^{n-1} G_{3t} + \dots + G_t^{n-1} A G_{3t} + G_t^n A_2). \quad (1.17)$$

Here, the 4-form $\sigma_4(A)$ coincides with the integrand of the functional (1.6). In general, a $(2n + 2)$ -form $\sigma_{2n+2}(A)$ is defined in $\mathcal{D} = 2n + 2 = 4, 6, 8, 10, \dots$ dimensions. The last equation is a generalization of the Chern-Simons density in $2n - 1$ dimensions (1.15). The dimensionality of this density is $[mass]^{n(n+2)}$, and it can be used as an addition to the $(2n+2)$ -dimensional Lagrangian density

$$\frac{1}{F^{n^2-2}} \int_{M_{2n+2}} \sigma_{2n+2}(A, A_2), \quad (1.18)$$

where F is a dimensional coupling constant, very similar to [21]. The $n = 1$ case $F \int_{M_4} \sigma_4$ was considered in [37] as a gauge invariant mass generation mechanism.

We also found a second series of exact $6n$ -forms constructed only in terms of the 3-form gauge field-strength G_3 :

$$\Delta_{6n} = Tr(G_3)^{2n} = d\pi_{6n-1}, \quad (1.19)$$

where for the $(6n - 1)$ -form one gets the following expression:

$$\pi_{6n-1}(A, A_2) = 2n \int_0^1 dt \, Tr(A_2 G_{3t}^{2n-1}). \quad (1.20)$$

These forms are defined in $\mathcal{D} = 6n - 1 = 5, 11, 17, \dots$ dimensions.

As it was well understood in [6, 7, 8, 9, 11, 13, 14, 17], the non-Abelian anomaly (1.13) is associated with ω_4^1 , the gauge variation of the density $\delta\omega_5 = d\omega_4^1$ in (1.15) and with ω_{2n-2}^1 in higher dimensions. Indeed, a celebrated result for the non-Abelian anomaly [6, 7, 8, 9, 11, 13, 14, 17] can be obtained by gauge variation of the ω_{2n-1} :

$$\delta\omega_{2n-1} = d\omega_{2n-2}^1, \quad (1.21)$$

where the $(2n - 2)$ -form has the following integral representation [6]:

$$\omega_{2n-2}^1(\xi, A) = n(n-1) \int_0^1 dt (1-t) \, Str(\xi d(A G_t^{n-2})), \quad (1.22)$$

where $\xi = \xi^a L_a$ is a scalar gauge parameter and Str denotes a symmetrized trace. In $\mathcal{D} = 2n - 2$ dimensions, the non-Abelian anomaly is given by this $(2n - 2)$ -form, the higher-dimensional analog of the equation (1.13):

$$D * J_\xi^{L,R} \propto \omega_{2n-2}^1(\xi, A). \quad (1.23)$$

Our next aim is to construct possible gauge anomalies σ_{2n+1}^1 and π_{6n-2}^1 which follow from the generalized densities σ_{2n+2} (1.17) and π_{6n-1} (1.20). These potential anomalies are defined through the relation analogous to (1.21):

$$\delta\sigma_{2n+2} = d\sigma_{2n+1}^1, \quad \delta\pi_{6n-1} = d\pi_{6n-2}^1. \quad (1.24)$$

The low-dimensional densities can be extracted directly from (1.10) and from (1.17). When we perform a vector-like gauge transformation ξ_1 , where $\xi_1 = \xi_\mu^a L_a dx^\mu$ is a 1-form gauge parameter (A.1), the corresponding densities are:

$$\sigma_3^1(\xi_1, A) = Tr(\xi_1 G), \quad \sigma_5^1(\xi_1, A) = Tr(\xi_1 d(AdA + \frac{1}{2}A^3)), \quad (1.25)$$

and when the gauge transformation is performed by a scalar gauge parameter ξ , then

$$\sigma_5^1(\xi, A, A_2) = Tr\left(\xi d(AdA_2 + A_2 dA + \frac{1}{2}A^2 A_2 - \frac{1}{2}AA_2 A + \frac{1}{2}A_2 A^2)\right), \quad (1.26)$$

What is interesting here is that σ_5^1 explicitly contains the second-rank gauge field A_2 when we perform the standard YM infinitesimal gauge transformation ξ . Because it is defined in odd dimensions it may have contribution to the parity-violating anomaly [52] and its descendant ($\delta\sigma_5^1 = d\sigma_4^2$)

$$\sigma_4^2(\xi, \eta, A) = \text{Tr}((d\xi \eta + \eta d\xi - \xi d\eta - d\eta \xi) dA_2) \quad (1.27)$$

may represent a potential Schwinger term in the corresponding gauge algebra [8, 9].

In the next section we shall present a short introduction into the theory of non-Abelian tensor gauge fields and discuss their small and large⁵ gauge transformations, their field-strength tensors, the corresponding flat connections of the tensor gauge fields and their Lagrangians [38, 39, 40]. In section 3, we shall derive the expression (1.9) for the large gauge transformation of the functional $\Sigma(A)$ and compare it with the corresponding transformation of $W(A)$. Equation (1.10) for the descendant density $\sigma_3^1(A, U)$ will be also presented. The corresponding integrands are the 4-form σ_4 (1.17) and 3-form σ_3^1 (1.25). In section 4, we present a topological invariant in six dimensions and its reduction to the 5-form π_5 and 4-form π_4^1 . In section 5, we shall derive general formulas for the σ_{2n+2} given in (1.17) and for the π_{6n-1} of eq. (1.20).

In conclusions, we shall discuss and compare the Pontryagin-Chern-Simons densities \mathcal{P}_{2n} , ω_{2n-1} and ω_{2n-2}^1 in YM gauge theory with the corresponding two series of densities Γ_{2n+3} , σ_{2n+2} , σ_{2n+1}^1 and Δ_{6n} , π_{6n-1} and π_{6n-2}^1 in the extended model containing non-Abelian tensor gauge fields. We shall also discuss different models suggested in the literature describing the dynamics of an antisymmetric non-Abelian tensor gauge field. The existing no-go theorem [44], which essentially limited possible non-Abelian models, can be circumvent only if the model contains infinitely many tensor gauge fields.

We have also included three appendices containing the basic formulas of tensor gauge fields that we use in the text and a short reminder on the winding number and non-Abelian anomaly.

2 *Small and Large Gauge Transformations*

Let us shortly overview the model of massless tensor gauge fields Lagrangian suggested in [38, 39, 40]. The gauge fields are defined as rank- $(s+1)$ tensors

$$A_{\mu\lambda_1\dots\lambda_s}^a(x),$$

⁵The transformations that are homotopic to the identity are called “small”, while those that cannot be deformed to the identity are called “large” [42, 11, 43].

which are totally symmetric with respect to the indices $\lambda_1 \dots \lambda_s$. The number of symmetric indices s runs from zero to infinity ⁶. The index a corresponds to the generators L_a of an appropriate Lie algebra. The extended non-Abelian gauge transformation δ_ξ (A.1), (A.2) of the tensor gauge fields is defined in the Appendix A and comprises a closed algebraic structure. The generalized field-strength tensors are defined as follows [38, 39, 40]:

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \\ G_{\mu\nu,\lambda} &= \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} - ig([A_\mu, A_{\nu\lambda}] + [A_{\mu\lambda}, A_\nu]), \\ G_{\mu\nu,\lambda\rho} &= \partial_\mu A_{\nu\lambda\rho} - \partial_\nu A_{\mu\lambda\rho} - ig([A_\mu, A_{\nu\lambda\rho}] + [A_{\mu\lambda}, A_{\nu\rho}] + [A_{\mu\rho}, A_{\nu\lambda}] + [A_{\mu\lambda\rho}, A_\nu]), \\ &\vdots \end{aligned} \quad (2.1)$$

and transform homogeneously with respect to the extended gauge transformations δ_ξ . The tensor gauge fields are in the matrix representation $A_{\mu\lambda_1\dots\lambda_s}^{ab} = (L_c)^{ab} A_{\mu\lambda_1\dots\lambda_s}^c = if^{acb} A_{\mu\lambda_1\dots\lambda_s}^c$ with f^{abc} - the structure constants of the Lie algebra.

Using field-strength tensors one can construct infinite series of forms \mathcal{L}_s invariant under the transformations δ_ξ . They are quadratic in field-strength tensors. The first terms are given by the formula [38, 39, 40]:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_2 + \dots = & - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a \\ & - \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a \\ & + \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + \frac{1}{4} G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + \frac{1}{2} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a + \dots \end{aligned} \quad (2.2)$$

The Lagrangian contains quadratic in gauge fields kinetic terms, as well as cubic and quartic terms describing non-linear interactions of gauge fields with dimensionless coupling constant g . The Lagrangian \mathcal{L} is well defined in any dimension.

In studying topological properties of the extending Yang-Mills theory it is important to define finite (not infinitesimal) gauge transformations of the tensor gauge fields. These can be found by expansion of the “large” transformation of the gauge field $\mathcal{A}_\mu(e)$ over the vector variable e^μ [40]. Thus the large gauge transformation of the tensor gauge fields takes the form

$$\begin{aligned} A_\mu^U &= U^- A_\mu U + \frac{i}{g} U^- \partial_\mu U, \\ A_{\mu\lambda}^U &= U^- A_{\mu\lambda} U + U^- A_\mu U_\lambda - U^- U_\lambda U^- A_\mu U + \frac{i}{g} (U^- \partial_\mu U_\lambda - U^- U_\lambda U^- \partial_\mu U), \\ &\vdots \end{aligned} \quad (2.3)$$

⁶ *A priori* the tensor fields have no symmetries with respect to the first index μ .

where U_λ is the second term in the expansion of the unitary matrix $\mathcal{U}(\Xi(x, e))$ over the vector variable:

$$\begin{aligned}\mathcal{U}(x, e) &= U(x) + U_\mu(x)e^\mu + \dots, \\ \mathcal{U}^-(x, e) &= U^-(x) - U^-(x)U_\mu(x)U^-(x)e^\mu + \dots\end{aligned}$$

The field-strength tensors transform correspondingly:

$$\begin{aligned}G_{\mu\nu}^U &= U^- G_{\mu\nu} U, \\ G_{\mu\nu,\lambda}^U &= U^- G_{\mu\nu,\lambda} U + U^- G_{\mu\nu} U_\lambda - U^- U_\lambda U^- G_{\mu\nu} U, \\ &\vdots\end{aligned}\tag{2.4}$$

One can obtain the expressions for the large gauge transformations of the higher-rank gauge fields $A_{\mu\lambda_1\dots\lambda_s}^a(x)$ by making further expansion of the unitary matrix $\mathcal{U}(\xi(x, e))$ over the vector variable e^μ . In order to recover the infinitesimal gauge transformations (A.1) and (A.2) of the tensor fields one should substitute the infinitesimal form of the matrices

$$U = 1 - igL_a \xi^a(x), \quad U_\mu = -igL_a \xi_\mu^a(x), \dots\tag{2.5}$$

into (2.3) and (2.4). As one can see, these matrix functions provide a mapping into the relevant gauge group G and into the corresponding algebra \mathcal{G} .

Let us now find *the flat connections, that is, the gauge field configurations which have non-trivial space-time behavior and for which the corresponding field-strength tensors (curvature) vanish*. The YM field-strength $G_{\mu\nu}$ vanishes when the vector potential is equal to a pure gauge connection:

$$A_\mu^{flat} = \frac{i}{g} U^- \partial_\mu U,\tag{2.6}$$

as it can be seen from the first equation in (2.3). The higher-rank field-strength tensor $G_{\mu\nu,\lambda}$ vanishes when the tensor field is equal to the last term of the second equation in (2.3):

$$A_{\mu\lambda}^{flat} = \frac{i}{g} (U^- \partial_\mu U_\lambda - U^- U_\lambda U^- \partial_\mu U).\tag{2.7}$$

It is therefore a “pure gauge connection” for the tensor gauge field. *A posteriori* one can become convinced that $G_{\mu\nu,\lambda}$ indeed vanishes by calculating the field-strength tensor (2.1) for the field configurations (2.6) and (2.7).

The physical states must be invariant under infinitesimal gauge transformation (2.5), or, equivalently, under finite gauge transformations that are continuously connected to the identity matrix $U = 1$ and to the zero vector matrices $U_\mu = 0$. But homotopically non-trivial gauge transformations that cannot be deformed to the identity or to the zero

vector matrices may also be present. The former can be called “small” in analogy with the standard YM theory [41, 42, 43, 13]. The gauge transformations that cannot be deformed to the identity or to the zero vector matrices are called “large”.

Let us find the explicit form of the vector matrices U_μ for the $SU(2)$ gauge group. Its group element $\mathcal{U}(\Xi(x, e))$ can be parameterized as follows:

$$\mathcal{U} = e^{-i\Xi^a \sigma^a} = \cos |\Xi| - i\hat{\Xi}^a \sigma^a \sin |\Xi|, \quad (2.8)$$

where $|\Xi| = \sqrt{\Xi^a \Xi^a}$, $\hat{\Xi}^a = \Xi^a / |\Xi|$ and σ^a are the Pauli matrices. Expanding the gauge parameter $\Xi^a(x, e)$ over the vector variable e^μ :

$$\Xi^a(x, e) = \xi^a(x) + \xi_\mu^a(x) e^\mu + \dots,$$

where $\xi^a(x)$ and $\xi_\mu^a(x)$ are space-time dependent gauge parameters, one can get the required matrices U_μ :

$$U_\mu = \frac{\partial \mathcal{U}}{\partial \Xi^a} \Big|_{e=0} \xi_\mu^a = -(\sin |\xi| + i\hat{\xi}^a \sigma^a \cos |\xi|) \hat{\xi}^b \xi_\mu^b - i\sigma^a (\delta^{ab} - \hat{\xi}^a \hat{\xi}^b) \xi_\mu^b \frac{\sin |\xi|}{|\xi|}, \quad (2.9)$$

as well as the other matrix combinations appearing in the previous expressions:

$$\begin{aligned} U^- U_\mu U^- &= (\sin |\xi| - i\hat{\xi}^a \sigma^a \cos |\xi|) \hat{\xi}^b \xi_\mu^b - i\sigma^a (\delta^{ab} - \hat{\xi}^a \hat{\xi}^b) \xi_\mu^b \frac{\sin |\xi|}{|\xi|}, \\ U_\mu U^- &= -i\hat{\xi}^a \sigma^a \hat{\xi}^b \xi_\mu^b - i\sigma^a (\delta^{ab} - \hat{\xi}^a \hat{\xi}^b) \xi_\mu^b (\cos |\xi| + i\hat{\xi}^c \sigma^c \sin |\xi|) \frac{\sin |\xi|}{|\xi|}, \\ U^- U_\mu &= -i\hat{\xi}^a \sigma^a \hat{\xi}^b \xi_\mu^b - i(\cos |\xi| + i\hat{\xi}^c \sigma^c \sin |\xi|) \sigma^a (\delta^{ab} - \hat{\xi}^a \hat{\xi}^b) \xi_\mu^b \frac{\sin |\xi|}{|\xi|}, \end{aligned} \quad (2.10)$$

where $|\xi| = \sqrt{\xi^a \xi^a}$, $\hat{\xi}^a = \xi^a / |\xi|$. As usual, non-trivial boundary conditions can be imposed by requiring the following asymptotic behavior:

$$|\xi| \xrightarrow{r \rightarrow \infty} > \pi N, \quad (2.11)$$

so that

$$U \xrightarrow{r \rightarrow \infty} > \pm 1, \quad U_\mu \xrightarrow{r \rightarrow \infty} > \mp i\hat{\xi}^a \sigma^a \hat{\xi}^b \xi_\mu^b. \quad (2.12)$$

3 Large Gauge Transformation of $\Sigma(A)$

The infinitesimal gauge transformations of $\Sigma(A)$ defined in (1.6) can be expressed as total derivative (1.8), so that its integral over four-dimensional space-time M_4 is given by contribution from the bounding surface and vanishes when the gauge parameter $\xi_\sigma(x)$ tends to zero sufficiently fast at the boundary. We are interested to know how $\Sigma(A)$

transforms when the gauge functions belong to a non-trivial homotopy class of unitary matrices.

From (2.6) and (2.7) it follows that for our lower-rank gauge fields *the variety of non-trivial flat connections are described by the matrices U and U_λ* . For a topological classification it is required that U tends to a constant at infinity. These gauge functions provide a mapping into the relevant gauge group G , and for non-Abelian compact gauge groups such mappings fall into disjoint homotopy classes labeled by an integer winding number $\Pi^3(G) = Z$ [18, 19, 20, 41, 42, 43, 13]. The analytic expression for the winding number can be found by considering how $W(A)$ of (1.7) transforms under large gauge transformations:

$$W(A^U) = W(A) + \frac{1}{8\pi^2} \int_{M_3} d^3x \varepsilon^{ijk} \partial_i \text{Tr} (\partial_j U U^{-1} A_k) + \frac{1}{24\pi^2} \int_{M_3} d^3x \varepsilon^{ijk} \text{Tr} (U^{-1} \partial_i U U^{-1} \partial_j U U^{-1} \partial_k U), \quad (3.1)$$

where the first term defines the non-Abelian anomaly in two-dimensions,

$$\omega_2^1(A, U) = \varepsilon^{ij} \text{Tr} (\partial_i U U^{-1} A_j), \quad (3.2)$$

which is the $n = 2$ case of the general expression for anomaly (1.22), and the last term is the winding number of the gauge function U (see Appendix B for a simple derivation)⁷.

There are many different ways to understand a topological character and the meaning of the functional $W(A)$. The above derivation is most suitable for our purposes. Indeed, in a similar way we would like to find out the transformation of the functional $\Sigma(A)$ when the gauge transformations are large. Using the expression for $\Sigma(A)$ given in (1.6) and the transformation laws (2.3) and (2.4) for the corresponding fields one can derive the transformation of the functional in the following form:

$$\Sigma(A^U) = \frac{1}{32\pi^2} \int_{M_4} d^4x \varepsilon^{\mu\nu\lambda\rho} \text{Tr} (U^{-1} G_{\mu\nu} U) (U^{-1} A_{\lambda\rho} U + U^{-1} A_\lambda U_\rho - U^{-1} U_\rho U^{-1} A_\lambda U + \frac{i}{g} (U^{-1} \partial_\lambda U_\rho - U^{-1} U_\rho U^{-1} \partial_\lambda U)).$$

Using cyclic permutation of the matrices under the trace Tr one can represent it as

$$\Sigma(A^U) = \Sigma(A) + \frac{1}{32\pi^2} \int_{M_4} d^4x \varepsilon^{\mu\nu\lambda\rho} \text{Tr} (G_{\mu\nu} [A_\lambda, U_\rho U^{-1}] + \frac{i}{g} G_{\mu\nu} \partial_\lambda (U_\rho U^{-1})), \quad (3.3)$$

⁷ The term $\frac{1}{8\pi^2} \int_{M_3} d^3x \varepsilon^{ijk} \partial_i \text{Tr} (\partial_j U U^{-1} A_k)$ does not contribute for vector potentials \vec{A} dropping sufficiently fast at infinity, $|\vec{A}| < 1/r$ [11, 43] and, in particular, for the pure gauge connections (2.6).

and then combining the last two terms into a covariant derivative we get

$$\begin{aligned}\Sigma(A^U) - \Sigma(A) &= \frac{i}{32\pi^2 g} \int_{M_4} d^4x \, \varepsilon^{\mu\nu\lambda\rho} \, \text{Tr} \{ G_{\mu\nu} \nabla_\lambda (U_\rho U^-) \} \\ &= \frac{i}{32\pi^2 g} \int_{M_4} d^4x \, \varepsilon^{\mu\nu\lambda\rho} \, \text{Tr} \{ \nabla_\lambda (G_{\mu\nu} U_\rho U^-) - (\nabla_\lambda G_{\mu\nu}) U_\rho U^- \}.\end{aligned}\quad (3.4)$$

By using Bianchi identity (A.3) the last expression can be reduced to the following boundary integral:

$$\begin{aligned}\Sigma(A^U) - \Sigma(A) &= \frac{i}{32\pi^2 g} \int_{M_4} d^4x \, \varepsilon^{\mu\nu\lambda\rho} \, \text{Tr} \{ \nabla_\lambda (G_{\mu\nu} U_\rho U^-) \} \\ &= \frac{i}{32\pi^2 g} \int_{M_4} d^4x \, \varepsilon^{\mu\nu\lambda\rho} \partial_\lambda \, \text{Tr} (G_{\mu\nu} U_\rho U^-) \\ &= \frac{i}{32\pi^2 g} \int_{\partial M_4} \varepsilon^{\mu\nu\lambda\rho} \, \text{Tr} (G_{\mu\nu} U_\rho U^-) \, d\sigma_\lambda.\end{aligned}\quad (3.5)$$

Thus the final expression for the large gauge transformation of $\Sigma(A)$ takes the form

$$\Sigma(A^U) - \Sigma(A) = \frac{i}{32\pi^2 g} \int_{M_4} d^4x \, \varepsilon^{\mu\nu\lambda\rho} \partial_\lambda \, \text{Tr} (G_{\mu\nu} U_\rho U^-). \quad (3.6)$$

The expression (3.6) is analogous to the corresponding formula for the Chern-Simons integral (3.1) and allows to introduce a lower-dimensional density

$$\sigma_3^1(A, U) = \varepsilon^{ijk} \, \text{Tr} (G_{ij} U_k U^-). \quad (3.7)$$

The gauge transformation of the $\sigma_3^1(A, U)$ does not generate a new density and the reduction stops. As we shall see below, the higher-dimensional density $\sigma_5^1(A, U)$ transgresses further to σ_4^2 (1.27).

4 Topological Density in Six Dimensions

In the previous sections we considered the densities in five and four dimensions. It is also possible to construct an invariant in six dimensions. Let us consider the following metric-independent density in six dimensions:

$$\Delta(A) = \varepsilon^{\mu\nu\lambda\rho\sigma\kappa} \, \text{Tr} \, G_{\mu\nu, \lambda} G_{\rho\sigma, \kappa}, \quad (4.1)$$

which is gauge invariant, because under infinitesimal gauge transformation δ_ξ (A.2) its variation vanishes:

$$\begin{aligned}\delta_\xi \Delta &= \varepsilon^{\mu\nu\lambda\rho\sigma\kappa} \text{Tr} (\delta G_{\mu\nu, \lambda} G_{\rho\sigma, \kappa} + G_{\mu\nu, \lambda} \delta G_{\rho\sigma, \kappa}) \\ &= -ig \varepsilon^{\mu\nu\lambda\rho\sigma\kappa} \text{Tr} ([G_{\mu\nu, \lambda} \xi] + [G_{\mu\nu} \xi_\lambda]) G_{\rho\sigma, \kappa} + G_{\mu\nu, \lambda} ([G_{\rho\sigma, \kappa} \xi] + [G_{\rho\sigma} \xi_\kappa]) = 0.\end{aligned}$$

Note that in the presence of a non-vanishing gravitational background $R_{\mu\nu\lambda\rho}$ the transformation of the field-strength tensors changes and instead of (A.2) has the following form:

$$\begin{aligned}\delta G_{\mu\nu} &= -ig[G_{\mu\nu} \xi], \\ \delta G_{\mu\nu,\lambda} &= -ig([G_{\mu\nu,\lambda} \xi] + [G_{\mu\nu} \xi_\lambda]) + 2R_{\mu\nu\lambda\rho}\xi^\rho, \\ &\vdots\end{aligned}\tag{4.2}$$

while $\Delta(A)$ remains gauge invariant in that case as well. The variation gets an additional term $\varepsilon^{\mu\nu\lambda\rho\sigma\kappa} Tr(G_{\mu\nu,\lambda} R_{\rho\sigma\kappa\alpha} \xi^\alpha)$, which is equal to zero if one uses the permutation property of the Riemann curvature tensor:

$$R_{\rho\sigma\kappa\alpha} + R_{\sigma\kappa\rho\alpha} + R_{\kappa\rho\sigma\alpha} = 0.$$

The density $\Delta(A)$ does not involve the space-time metric and is diffeomorphism invariant. Moreover, it is invariant not only under infinitesimal gauge transformations δ_ξ , but also under large transformations (2.4):

$$\Delta(A^U) = \Delta(A).\tag{4.3}$$

The density $\Delta(A)$ is a total derivative of a vector current Π_μ :

$$\Delta(A) = \varepsilon^{\mu\nu\lambda\rho\sigma\kappa} Tr G_{\mu\nu,\lambda} G_{\rho\sigma,\kappa} = 2 \partial_\mu \Pi^\mu,\tag{4.4}$$

where

$$\Pi^\mu(A) = \varepsilon^{\mu\nu\lambda\rho\sigma\kappa} Tr G_{\nu\lambda,\rho} A_{\sigma\kappa}.\tag{4.5}$$

While $\Delta(A)$ and the vector current $\Pi_\mu(A)$ are defined on a six-dimensional manifold, we may restrict the latter to a lower, five-dimensional manifold. Considering, indeed, the sixth component of the vector current Π_μ

$$\varepsilon^{5\nu\lambda\rho\sigma\kappa} Tr G_{\nu\lambda,\rho} A_{\sigma\kappa}.\tag{4.6}$$

one sees that the remaining indices do not repeat the external index and the sum is restricted to five-dimensional indices. One can thus reduce this functional to five dimensions, considering gauge fields independent of the sixth coordinate x_5 . The density Π^5 is then well defined in five-dimensional space-time and, as we shall see, it is also gauge invariant up to a total divergence term. We can therefore consider its integral over five-dimensional space-time:

$$\Pi(A) = \varepsilon^{\nu\lambda\rho\sigma\kappa} \int_{M_5} d^5x Tr G_{\nu\lambda,\rho} A_{\sigma\kappa}.\tag{4.7}$$

This functional is gauge invariant up to a total divergence term. Its infinitesimal gauge variation under δ_ξ (A.1)-(A.2) is given by

$$\delta_\xi \int_{M_5} d^5x \Pi = \varepsilon^{\nu\lambda\rho\sigma\kappa} \int_{M_5} \partial_\sigma \text{Tr}(G_{\nu\lambda,\rho} \xi_\kappa) d^5x = \varepsilon^{\nu\lambda\rho\sigma\kappa} \int_{\partial M_5} \text{Tr}(G_{\nu\lambda,\rho} \xi_\kappa) d\sigma_\sigma = 0,$$

where the boundary term vanishes when the gauge parameter ξ_κ tends to zero at infinity.

But it changes under large gauge transformations. Let us calculate its variation.

Using the transformation properties (2.3) and (2.4) of the corresponding fields, one can derive the transformation of the functional in the following form:

$$\begin{aligned} \Pi(A^U) &= \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} (U^- G_{\mu\nu,\lambda} U + U^- G_{\mu\nu} U_\lambda - U^- U_\lambda U^- G_{\mu\nu} U) \\ &\quad (U^- A_{\rho\sigma} U + U^- A_\rho U_\sigma - U^- U_\sigma U^- A_\rho U + \frac{i}{g} (U^- \partial_\rho U_\sigma - U^- U_\sigma U^- \partial_\rho U)). \end{aligned}$$

Using cyclic permutation of the matrices under the trace, one can represent it as

$$\begin{aligned} \Pi(A^U) &= \Pi(A) + \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} \{ G_{\mu\nu,\lambda} [A_\rho, U_\sigma U^-] + \frac{i}{g} G_{\mu\nu,\lambda} \partial_\rho (U_\sigma U^-) + \\ &\quad + G_{\mu\nu} U_\lambda U^- [A_\rho, U_\sigma U^-] + \frac{i}{g} G_{\mu\nu} U_\lambda U^- (\partial_\rho U_\sigma U^-) - \\ &\quad - G_{\mu\nu} [A_\rho, U_\sigma U^-] U_\lambda U^- - \frac{i}{g} G_{\mu\nu} (\partial_\rho U_\sigma U^-) U_\lambda U^- \\ &\quad + G_{\mu\nu} [U_\lambda U^- A_{\rho\sigma}] - \frac{i}{g} G_{\mu\nu} (\partial_\rho U_\sigma U^-) U_\lambda U^- + [A_{\rho\sigma} G_{\mu\nu}] U_\lambda U^- \} \end{aligned}$$

and then, combining the terms into the covariant derivative, we get

$$\begin{aligned} \Pi(A^U) - \Pi(A) &= \\ &= \frac{i}{g} \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} \{ G_{\mu\nu,\lambda} \nabla_\rho (U_\sigma U^-) - ig [A_{\rho\sigma} G_{\mu\nu}] U_\lambda U^- + G_{\mu\nu} \nabla_\rho (U_\lambda U^- U_\sigma U^-) \} \\ &= \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} \{ \nabla_\rho (G_{\mu\nu,\lambda} U_\sigma U^-) - (\nabla_\rho G_{\mu\nu,\lambda}) U_\sigma U^- + ig [A_{\rho\lambda} G_{\mu\nu}] U_\sigma U^- + \\ &\quad + \nabla_\rho (G_{\mu\nu} U_\lambda U^- U_\sigma U^-) - (\nabla_\rho G_{\mu\nu}) U_\lambda U^- U_\sigma U^- \}. \end{aligned}$$

By using Bianchi identity (A.4) the last expression can be reduced to the following boundary integral:

$$\begin{aligned} \Pi(A^U) - \Pi(A) &= \frac{i}{g} \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} \{ \nabla_\rho (G_{\mu\nu,\lambda} U_\sigma U^- + G_{\mu\nu} U_\lambda U^- U_\sigma U^-) \} \\ &= \frac{i}{g} \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \partial_\rho \text{Tr} (G_{\mu\nu,\lambda} U_\sigma U^- + G_{\mu\nu} U_\lambda U^- U_\sigma U^-) \\ &= \frac{i}{g} \int_{\partial M_5} \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} (G_{\mu\nu,\lambda} U_\sigma U^- + G_{\mu\nu} U_\lambda U^- U_\sigma U^-) d\sigma_\rho. \end{aligned}$$

Thus, the large gauge transformation of $\Pi(A)$ is

$$\Pi(A^U) - \Pi(A) = \frac{i}{g} \int_{M_5} d^5x \varepsilon^{\mu\nu\lambda\rho\sigma} \partial_\rho \text{Tr} (G_{\mu\nu,\lambda} U_\sigma U^- + G_{\mu\nu} U_\lambda U^- U_\sigma U^-). \quad (4.8)$$

The expression (4.8) is analogous to the corresponding formula for the Chern-Simons integral (3.1) and to our previous result obtained for the $\Sigma(A^U) - \Sigma(A)$ in (3.6). We can introduce now a lower-dimensional functional representing potential anomaly in four-dimensions:

$$\pi_4^1(A, U) = \varepsilon^{\mu\nu\lambda\rho} \text{Tr} (G_{\mu\nu, \lambda} U_\sigma U^-) . \quad (4.9)$$

The dimensional reduction of the variation $\delta\Sigma$ to $\sigma_3^1(A, U)$ of (3.7) and of $\delta\Pi$ to $\pi_4^1(A, U)$ of (4.9) stops because a gauge transformation of the last densities leaves their form unchanged.

5 *Anomalies and Densities in Higher Dimensions*

As we already discussed in the introduction, all chiral anomalies, Abelian and non-Abelian, can be determined by a differential geometric method without having to evaluate a Feynman diagram [6, 7, 8, 11, 12, 13, 14, 17, 45, 46, 47, 48, 49, 50, 51]. The non-Abelian anomaly in $2n$ -dimensional space-time may be obtained from the Abelian anomaly in $2n + 2$ dimensions by a reduction (transgression) procedure [6, 7, 8, 11, 12, 13, 14, 17]. The $U_A(1)$ anomaly appears in the divergence of the axial $U(1)$ current $J_\mu^A = \bar{\psi}\gamma_\mu\gamma_5\psi$ and in four-dimensional space-time it is given by

$$\partial^\mu J_\mu^A = -\frac{1}{16\pi^2} \varepsilon^{\mu\nu\lambda\rho} \text{Tr}(G_{\mu\nu} G_{\lambda\rho}) = -\frac{1}{4\pi^2} \varepsilon^{\mu\nu\lambda\rho} \partial_\mu \text{Tr}(A_\nu \partial_\lambda A_\rho - i\frac{2}{3} g A_\nu A_\lambda A_\rho). \quad (5.1)$$

The non-Abelian anomaly appears in the covariant divergence of the non-Abelian left- and right-handed currents $J_\mu^{aL} = \bar{\psi}_L \gamma_\mu \gamma_5 \sigma^a \psi_L$, $J_\mu^{aR} = \bar{\psi}_R \gamma_\mu \gamma_5 \sigma^a \psi_R$ with

$$D^\mu J_\mu^{aL} = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} \partial_\mu \text{Tr}[\sigma^a (A_\nu \partial_\lambda A_\rho - i\frac{1}{2} g A_\nu A_\lambda A_\rho)]. \quad (5.2)$$

The Abelian anomaly is gauge invariant, while the non-Abelian anomaly is gauge covariant and is given by the covariant divergence of a non-Abelian current.

In order to introduce higher-dimensional densities it is convenient to use the language of forms [6, 7, 14]. We already introduced the one- and two-form gauge potentials $A = -ig A_\mu^a L_a dx^\mu$ and $A_2 = -ig A_{\mu\nu}^a L_a dx^\mu dx^\nu$ with the corresponding field-strength tensors (2.1):

$$G = dA + A^2, \quad G_3 = dA_2 + [A, A_2]. \quad (5.3)$$

The Bianchi identities (A.3) and (A.4) now take the form

$$DG = 0, \quad DG_3 + [A_2, G] = 0, \quad (5.4)$$

where $DG = dG + [A, G]$ and $DG_3 = dG_3 + AG_3 + G_3A$. Let us consider a higher-dimensional invariant density in $2n + 3$ space-time dimensions:

$$\Gamma_{2n+3} = Tr(G^n G_3), \quad (5.5)$$

which coincides with $\Gamma(A)$ of (1.1) for $n = 1$ and is a natural generalization of the Chern-Pontryagin form $\mathcal{P}_{2n} = Tr(G^n)$. By direct computation of the derivative one can prove that Γ_{2n+3} is an exact form:

$$\begin{aligned} d\Gamma_{2n+3} &= Tr(dGG^{n-1}G_3 + \dots + G^{n-1}dGG_3 + G^n dG_3) \\ &= Tr((dG + [A, G])G^{n-1}G_3 + \dots + G^n(dG_3 + AG_3 + G_3A)) \\ &= Tr(DGG^{n-1}G_3 + \dots + G^{n-1}DGG_3 + G^n DG_3) \\ &= Tr(G^n DG_3) = Tr(G^n(DG_3 + [A_2, G])) = 0. \end{aligned}$$

In this calculation one must change sign when transmitting the differential d through an odd form or commuting odd forms using the cyclic property of the trace, and use Bianchi identities as well. According to Poincaré's lemma, this equation implies that Γ_{2n+3} can be locally written as an exterior differential of a certain $(2n+2)$ -form. In order to find the form of which Γ_{2n+3} is the derivative we have to find its variation, induced by the variation of the fields δA and δA_2 :

$$\delta G = D(\delta A), \quad \delta G_3 = D(\delta A_2) + [\delta A, A_2]$$

yielding a variation of Γ_{2n+3} which is a total derivative:

$$\delta\Gamma_{2n+3} = d Tr(\delta AG^{n-1}G_3 + \dots + G^{n-1}\delta AG_3 + G^n \delta A_2). \quad (5.6)$$

Following [6], we introduce a one-parameter family of potentials and strengths through the parameter t ($0 \leq t \leq 1$):

$$A_t = tA, \quad G_t = tG + (t^2 - t)A^2, \quad A_{2t} = tA_2, \quad G_{3t} = tG_3 + (t^2 - t)[A, A_2], \quad (5.7)$$

so that the equation (5.6) can be rewritten as

$$\delta Tr(G_t^n G_{3t}) = d Tr(\delta A_t G_t^{n-1} G_{3t} + \dots + G_t^{n-1} \delta A_t G_{3t} + G_t^n \delta A_{2t}).$$

With $\delta = \delta t(\partial/\partial t)$ and $\delta A_t = A\delta t$, $\delta A_{2t} = A_2\delta t$ we shall get by integration the desired result:

$$Tr(G^n G_3) = d \sigma_{2n+2}, \quad (5.8)$$

where the corresponding secondary $(2n + 2)$ -form is

$$\sigma_{2n+2}(A, A_2) = \int_0^1 dt \operatorname{Tr}(AG_t^{n-1}G_{3t} + \dots + G_t^{n-1}AG_{3t} + G_t^n A_2). \quad (5.9)$$

The dimensionality of this density is $[mass]^{n(n+2)}$ and it can be used as an addition to the $(2n+2)$ -dimensional Lagrangian density [37]

$$\frac{1}{F^{n^2-2}} \int_{M_{2n+2}} \sigma_{2n+2}(A, A_2), \quad (5.10)$$

where F is a dimensional coupling constant. The $n = 1$ case was considered in [37]. It has the form:

$$F \int_{M_4} \sigma_4(A, A_2)$$

and can be added to the Lagrangian density generating masses of the vector gauge bosons. The equation (5.9) is a generalization of the formula for the n -th Chern-Pontryagin character and of the corresponding Chern-Simons secondary topological invariant $(2n - 1)$ -form [6, 7, 13]⁸:

$$\operatorname{Tr}(G^n) = d \omega_{2n-1}, \quad (5.11)$$

where

$$\omega_{2n-1}(A) = n \int_0^1 dt \operatorname{Tr}(AG_t^{n-1}). \quad (5.12)$$

In $\mathcal{D} = 2n$ dimensions, the $U_A(1)$ anomaly is given by this $2n$ -form, the higher-dimensional analog of eq. (5.1):

$$d * J^A \propto \operatorname{Tr}(G^n) = d \omega_{2n-1}.$$

Note that the significance of the densities (5.9) in extended YM theory and their connection with anomalies in different dimensions is yet to be understood.

Returning back to the expression (5.9), for $n = 1$ one can recover the expressions (1.1) and (1.2) in four dimensions:

$$\sigma_4 = \int_0^1 dt \operatorname{Tr}(AG_{3t} + G_t A_2) = \operatorname{Tr}(GA_2) \quad (5.13)$$

In six dimensions, $n = 2$, we have

$$\sigma_6 = \int_0^1 dt \operatorname{Tr}(AG_t G_{3t} + G_t AG_{3t} + G_t^2 A_2),$$

and after integration over t we get a secondary 6-form:

$$\begin{aligned} \sigma_6(A, G, G_3) &= \frac{1}{3} \operatorname{Tr}(AGG_3 + AG_3G + A_2G^2 - \frac{1}{2}A^3G_3 \\ &\quad - \frac{1}{2}(A^2A_2 - AA_2A + A_2A^2)G + \frac{1}{2}A^4A_2). \end{aligned} \quad (5.14)$$

⁸The derivation can also be found in Appendix C.

The new property of the last functional compared with σ_4 above is that when the field-strength tensors tend to zero, $G = G_3 = 0$, the functional does not vanish and is equal to

$$\frac{1}{6} \text{Tr}(A^4 A_2), \quad (5.15)$$

where one should substitute the flat connections (2.6) and (2.7). The subsequent forms σ_{2n+2} are in $\mathcal{D} = 2n + 2 = 4, 6, 8, 10, \dots$ dimensions. The form (5.14) is the analog of the Chern-Simons (CS) 5-form [13]

$$\omega_5(A, G) = \text{Tr}(AG^2 - \frac{1}{2}A^3G + \frac{1}{10}A^5),$$

and (5.15) is the analog of the winding number $\text{Tr}(A^5)$ in five dimensions and of the corresponding Wess-Zumino-Witten chiral effective action [21].

The second series of invariant forms can be constructed by generalization of the density (4.1) to higher dimensions. It can be written as follows:

$$\Delta_{6n} = \text{Tr}(G_3)^{2n}, \quad (5.16)$$

and is an exact $6n$ -form:

$$\begin{aligned} d\Delta_{6n} &= n \text{Tr} G_3^{2n-2} (dG_3 G_3 - G_3 dG_3) = n \text{Tr} G_3^{2n-2} (DG_3 G_3 - G_3 DG_3) \\ &= n \text{Tr} G_3^{2n-2} ((dG_3 + [A_2 G])G_3 - G_3(dG_3 + [A_2, G])) = 0. \end{aligned} \quad (5.17)$$

Its variation over the gauge fields is

$$\delta\Delta_{6n} = 2n d \text{Tr}(\delta A_2 G_3^{2n-1}) = d\pi_{6n-1}, \quad (5.18)$$

so that after introducing the t deformation of the fields we get the $(6n - 1)$ -form:

$$\pi_{6n-1}(A, A_2) = 2n \int_0^1 dt \text{Tr}(A_2 G_{3t}^{2n-1}). \quad (5.19)$$

For $n = 1$ it reproduces the 5-form (4.7)

$$\pi_5 = 2 \int_0^1 dt \text{Tr}(A_2 G_{3t}) = \text{Tr}(A_2 G_3) \quad (5.20)$$

and for $n = 2$ we get the 11-form

$$\pi_{11} = 4 \int_0^1 dt \text{Tr} A_2 (tG_3 + (t^2 - t)[A, A_2])^3, \quad (5.21)$$

which after integration yields

$$\begin{aligned} \pi_{11} &= \text{Tr} \left(A_2 (G_3^3 - \frac{1}{5}([A, A_2]G_3^2 + G_3[A, A_2]G_3 + G_3^2[A, A_2])) + \right. \\ &\quad \left. + \frac{1}{15}([A, A_2]^2 G_3 + [A, A_2]G_3[A, A_2] + G_3[A, A_2]^2) - \frac{1}{35}[A, A_2]^3 \right). \end{aligned} \quad (5.22)$$

The forms π_{6n-1} are defined in $\mathcal{D} = 6n - 1 = 5, 11, 17, \dots$ dimensions.

Now, having in hand the $(2n+2)$ -form σ_{2n+2} (5.9) and $(6n-1)$ -form π_{6n-1} (5.19), we can turn to the construction of possible anomalies in extended YM theory. Indeed, the non-Abelian anomalies can be determined through the geometric procedure as follows. From the gauge invariance of the densities Γ_{2n+3} and Δ_{6n} and from (5.8) and (5.16) we know that

$$\begin{aligned}\delta \Gamma_{2n+3} &= \delta d \sigma_{2n+2} = d \delta \sigma_{2n+2} = 0, \\ \delta \Delta_{6n} &= \delta d \pi_{6n-1} = d \delta \pi_{6n-1} = 0.\end{aligned}$$

Thus locally there must exist a certain $(2n+1)$ -form $\sigma_{2n+1}^1(\xi, A)$ and $(6n-2)$ -form $\pi_{6n-2}^1(\xi, A)$ such that

$$\delta \sigma_{2n+2}(A) = d\sigma_{2n+1}^1(\xi, A), \quad \delta \pi_{6n-1}(A) = d\pi_{6n-2}^1(\xi, A). \quad (5.23)$$

Here the superscript of $\sigma_{2n+1}^1(\xi, A)$ and $\pi_{6n-2}^1(\xi, A)$ indicates that these forms are of first order in the gauge parameters.

These anomalies for lower dimensions are already known through the calculations we made in the previous sections. Indeed, we calculated the global gauge variation of the secondary forms σ_4 and π_5 and found that they are total derivatives, thus from (3.7) for infinitesimal gauge transformations (2.5) it follows that

$$\sigma_3^1(\xi_1, A) = \text{Tr}(\xi_1 G), \quad (5.24)$$

where ξ_1 is a 1-form gauge parameter $\xi_1 = L^a \xi_\mu^a dx^\mu$, and from (4.9) we can extract π_4^1 :

$$\pi_4^1(\xi_1, A) = \text{Tr}(\xi_1 G_3). \quad (5.25)$$

In both cases there is no dependence from the scalar gauge parameter $\xi = L^a \xi^a$.

In order to calculate the variation of the secondary characteristics in higher dimensions we need the formulas for the gauge transformation of the various fields involved in the expressions for σ_{2n+2} (5.9) and π_{6n-1} (5.19), which read

$$\begin{aligned}\delta_\xi A &= d\xi + [A, \xi], & \delta_\xi A_2 &= d\xi_1 + A\xi_1 + \xi_1 A + [A_2, \xi], \\ \delta_\xi dA &= [dA, \xi] - Ad\xi - d\xi A, & \delta_\xi dA_2 &= [dA, \xi_1] - [A, d\xi_1] + [dA_2, \xi] + [A_2, d\xi], \\ \delta_\xi G_t &= [G_t, \xi] + (t^2 - t)(Ad\xi + d\xi A), \\ \delta_\xi G_{3t} &= [G_{3t}, \xi] + [G_t, \xi_1] + (t^2 - t)([A, d\xi_1] + [A_2, d\xi]).\end{aligned} \quad (5.26)$$

Let us calculate the variation of σ_6 (5.14) using the above formulas. It turns out that there are many cancelations between different terms, so that at the end we get two contributions,

one linear in $d\xi$ and the other - in $d\xi_1$. The term linear in the differential of the 1-form gauge parameter $d\xi_1$ is

$$\begin{aligned}\delta\sigma_6 &= \text{Tr}(d\xi_1 G^2 - \frac{1}{2}A^2 d\xi_1 G + \frac{1}{2}Ad\xi_1 AG - \frac{1}{2}d\xi_1 A^2 G + \frac{1}{2}A^4 d\xi_1) = \\ &= \text{Tr}(d\xi_1 d(AdA + \frac{1}{2}A^3)) = d\sigma_5^1,\end{aligned}$$

therefore

$$\sigma_5^1(\xi_1, A) = \text{Tr}(\xi_1 d(AdA + \frac{1}{2}A^3)). \quad (5.27)$$

This expression coincides with the standard gauge anomaly in four dimensions ω_4^1 (5.2), with the only difference that it is multiplied by ξ_1 , which is here a 1-form. The second term linear in $d\xi$ is

$$\begin{aligned}\delta\sigma_6 &= \text{Tr}(d\xi GG_3 + d\xi G_3 G - \frac{1}{2}(d\xi A^2 G_3 + Ad\xi AG_3 + A^2 d\xi G_3) - \\ &- \frac{1}{2}(d\xi AA_2 G + Ad\xi A_2 G - d\xi A_2 AG - AA_2 d\xi G + A_2 d\xi AG + A_2 Ad\xi G) + \\ &+ \frac{1}{2}(d\xi A^3 A_2 + Ad\xi A^2 A_2 + A^2 d\xi AA_2 + A^3 d\xi A_2) = \\ &= \text{Tr}\left(d\xi d(AdA_2 + A_2 dA + \frac{1}{2}(A^2 A_2 - AA_2 A + A_2 A^2))\right) = d\sigma_5^1,\end{aligned}$$

and therefore

$$\begin{aligned}\sigma_5^1(\xi, A, A_2) &= \text{Tr}\left(\xi d(AdA_2 + A_2 dA + \frac{1}{2}(A^2 A_2 - AA_2 A + A_2 A^2))\right) \\ &= \text{Tr}\left(\xi d(AG_3 + A_2 G - \frac{1}{2}(A^2 A_2 - AA_2 A + A_2 A^2))\right). \quad (5.28)\end{aligned}$$

The total form σ_5^1 is a sum of the two expressions above, (5.27) and (5.28). What is interesting here is that σ_5^1 explicitly contains the second-rank gauge field A_2 when we perform the standard YM infinitesimal gauge transformation ξ . Because it is defined in odd dimensions it may have contribution to the parity-violating anomaly [52] and its descendant ($\delta\sigma_5^1 = d\sigma_4^2$):

$$\sigma_4^2(\xi, \eta, A) = \text{Tr}((d\xi \eta + \eta d\xi - \xi d\eta - d\eta \xi) dA_2) \quad (5.29)$$

may represent a potential anomalous Schwinger term in the corresponding gauge algebras [8]. Here the superscript of $\sigma_4^2(\xi, \eta, A)$ indicates that this form is of second order in the gauge parameters.

6 Conclusions

In conclusion let us compare the Pontryagin-Chern-Simons densities \mathcal{P}_{2n} , ω_{2n-1} and ω_{2n-2}^1 in YM gauge theory with the corresponding densities Γ_{2n+3} , σ_{2n+2} , σ_{2n+1}^1 and Δ_{6n} , π_{6n-1} ,

π_{6n-2}^1 in the extended YM theory. The new characteristic classes are local forms defined on the space-time manifold and constructed from the curvature 2-form G and 3-form G_3 :

$$\Gamma_{2n+3} = Tr(G^n G_3) = d \sigma_{2n+2} , \quad \Delta_{6n} = Tr(G_3)^{2n} = d \pi_{6n-1} . \quad (6.1)$$

These characteristic classes are closed forms, but not globally exact. The secondary characteristic classes σ_{2n+2} and π_{6n-1} can be expressed in integral form (5.9) and (5.19) in analogy with the Chern-Simons form (5.12). Their gauge variation can also be found, yielding the potential anomalies in gauge field theory. The above general considerations should be supplemented by an explicit calculation of loop diagrams involving chiral fermions. The argument in favor of the existence of these potential anomalies is based on the fact that they fulfill Wess-Zumino consistency conditions. At the same time, these invariant densities constructed on the space-time manifold have their own independent value since they suggest the existence of new invariants characterizing topological properties of a manifold.

The Abelian version of the invariant $\Sigma(A)$ was investigated earlier in [28, 29, 30, 31, 32, 33, 34, 35, 53, 54]. Attempts to construct a non-Abelian invariant in a similar way have come up with difficulties because they involve non-Abelian generalization of gauge transformations of antisymmetric fields [44, 55, 56]. The no-go theorem [44] implies that without additional auxiliary fields the gauge transformations cannot form a closed group. And, indeed, the gauge transformations of non-Abelian tensor gauge fields δ_ξ (A.1) cannot be limited to a YM 1-form and rank-2 antisymmetric field. Instead, the antisymmetric tensor is extended by a symmetric rank-2 gauge field, so that together they form a gauge field $A_{\mu\nu}^a$ which transforms according to (A.1) and is a fully propagating field. It is also important that one should include all high-rank gauge fields $A_{\mu\lambda_1\dots\lambda_s}^a$ in order to be able to close the group of gauge transformations and to construct an invariant Lagrangian.

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A Tensor gauge fields

The extended non-Abelian gauge transformation δ_ξ of tensor gauge fields is defined by the equations [38, 39, 40]:

$$\begin{aligned}\delta_\xi A_\mu &= \partial_\mu \xi - ig[A_\mu, \xi] \\ \delta_\xi A_{\mu\nu} &= \partial_\mu \xi_\nu - ig[A_\mu, \xi_\nu] - ig[A_{\mu\nu}, \xi] \\ \delta_\xi A_{\mu\nu\lambda} &= \partial_\mu \xi_{\nu\lambda} - ig[A_\mu, \xi_{\nu\lambda}] - ig[A_{\mu\nu}, \xi_\lambda] - ig[A_{\mu\lambda}, \xi_\nu] - ig[A_{\mu\nu\lambda}, \xi], \\ &\vdots\end{aligned}\tag{A.1}$$

where $\xi_{\lambda_1 \dots \lambda_s}^a(x)$ are totally symmetric gauge parameters, and comprises a closed algebraic structure. The tensor gauge fields are in the matrix representation $A_{\mu\lambda_1 \dots \lambda_s}^{ab} = (L_c)^{ab} A_{\mu\lambda_1 \dots \lambda_s}^c = if^{acb} A_{\mu\lambda_1 \dots \lambda_s}^c$ with f^{abc} - the structure constants. The generalized field-strength tensors (2.1) transform homogeneously under the extended gauge transformations δ_ξ :

$$\begin{aligned}\delta G_{\mu\nu}^a &= -ig[G_{\mu\nu}, \xi], \\ \delta G_{\mu\nu,\lambda}^a &= -ig([G_{\mu\nu,\lambda}, \xi] + [G_{\mu\nu}, \xi_\lambda]), \\ \delta G_{\mu\nu,\lambda\rho}^a &= -ig([G_{\mu\nu,\lambda\rho}^b, \xi] + [G_{\mu\nu,\lambda}, \xi_\rho] + [G_{\mu\nu,\rho}, \xi_\lambda] + [G_{\mu\nu}, \xi_{\lambda\rho}]), \\ &\vdots\end{aligned}\tag{A.2}$$

In the YM theory the Bianchi identity is

$$[\nabla_\mu, G_{\nu\lambda}] + [\nabla_\nu, G_{\lambda\mu}] + [\nabla_\lambda, G_{\mu\nu}] = 0,\tag{A.3}$$

and for the higher-rank field-strength tensors $G_{\nu\lambda,\rho}$ and $G_{\nu\lambda,\rho\sigma}$ the Bianchi identities are:

$$[\nabla_\mu, G_{\nu\lambda,\rho}] - ig[A_{\mu\rho}, G_{\nu\lambda}] + [\nabla_\nu, G_{\lambda\mu,\rho}] - ig[A_{\nu\rho}, G_{\lambda\mu}] + [\nabla_\lambda, G_{\mu\nu,\rho}] - ig[A_{\lambda\rho}, G_{\mu\nu}] = 0,\tag{A.4}$$

$$[\nabla_\mu, G_{\nu\lambda,\rho\sigma}] - ig[A_{\mu\rho}, G_{\nu\lambda,\sigma}] - ig[A_{\mu\sigma}, G_{\nu\lambda,\rho}] - ig[A_{\mu\rho\sigma}, G_{\nu\lambda}] + cyc.perm.(\mu\nu\lambda) = 0\tag{A.5}$$

and so on.

B Winding Number

The variation of the CS term under large gauge transformation can be computed by using parametrization (2.8) for the SU(2) group elements [41, 42, 43]. We used the mathematica program "Exterior Differential Calculus" developed by Sotirios Bonanos [57] to evaluate

the wedge products

$$\begin{aligned}
W(A^U) - W(A) &= \frac{1}{24\pi^2} \int_{M_3} d^3x \, \varepsilon^{ijk} \, \text{Tr} \, (U^- \partial_i U \, U^- \partial_j U \, U^- \partial_k U) \\
&= \frac{1}{24\pi^2} \int_{M_3} \text{Tr} \, (U^- dU \wedge U^- dU \wedge U^- dU) = \\
&= \frac{1}{2\pi^2} \int_{M_3} \frac{\sin^2 |\xi|}{|\xi|^2} d\xi_1 \wedge d\xi_2 \wedge d\xi_3 = \\
&= -\frac{1}{16\pi^2} \int_{M_3} \varepsilon^{abc} d(\hat{\xi}^a \hat{\xi}^b \wedge d\hat{\xi}^c (\sin 2|\xi| - 2|\xi|)) = \\
&= -\frac{1}{16\pi^2} \int_{\partial M_3} \varepsilon^{abc} \hat{\xi}^a \hat{\xi}^b \wedge d\hat{\xi}^c (\sin 2|\xi| - 2|\xi|) = \\
&= -\frac{1}{8\pi^2} \int_{\partial M_3} (\sin 2|\xi| - 2|\xi|) \hat{\xi}^a d\omega^a, \tag{B.1}
\end{aligned}$$

where $d\omega^a = 2\varepsilon^{abc} d\hat{\xi}^b \wedge d\hat{\xi}^c$. With the boundary condition $|\xi| \xrightarrow{r \rightarrow \infty} > \pi N$ one gets

$$W(A^U) - W(A) = \frac{1}{4\pi^2} \int_{\partial M_3} |\xi| \hat{\xi}^a d\omega^a = N. \tag{B.2}$$

C Appendix C. Non-Abelian Anomaly

As an exercise let us calculate the non-Abelian anomaly by gauge variation of ω_{2n-1} in YM theory (5.12) [6, 7, 8, 11, 12, 13, 14]. Using formulas (5.26) one can get

$$\begin{aligned}
\delta_\xi \omega_{2n-1}(A, G) &= n \int_0^1 dt \, \delta_\xi \, \text{Str}(AG_t^{n-1}) = \\
&= n \int_0^1 dt \, \text{Str}((d\xi + [A, \xi])G_t^{n-1} + A([G_t, \xi] + (t^2 - t)\{A, d\xi\})G_t^{n-2} + \\
&+ AG_t([G_t, \xi] + (t^2 - t)\{A, d\xi\})G_t^{n-3} + \dots + AG_t^{n-2}([G_t, \xi] + (t^2 - t)\{A, d\xi\})).
\end{aligned}$$

Collecting terms which contain $d\xi$ and ξ into two separate integrals we get

$$\begin{aligned}
&n \int_0^1 dt \, \text{Str}(d\xi G_t^{n-1} + (t^2 - t)A(\{A, d\xi\}G_t^{n-2} + G_t\{A, d\xi\}G_t^{n-3} + \dots + G_t^{n-2}\{A, d\xi\})) \\
&+ n \int_0^1 dt \, \text{Str}([A, \xi]G_t^{n-1} + A[G_t, \xi]G_t^{n-2} + AG_t[G_t, \xi]G_t^{n-3} + \dots + AG_t^{n-2}[G_t, \xi]).
\end{aligned}$$

Opening the brackets in the second integral one can see that it vanishes, while the first integral can be represented as

$$n \int_0^1 dt \, \text{Str}(d\xi G_t^{n-1} + (t^2 - t)(n-1)(\{A, A\}d\xi G_t^{n-2} + Ad\xi[A, G_t^{n-2}]),$$

and by using the equations

$$dG_t^{n-2} = -[A_t, G_t^{n-2}], \quad \frac{\partial G_t}{\partial t} = dA + t\{A, A\}$$

it can be rewritten as

$$n \int_0^1 dt \operatorname{Str} \left(d\xi G_t^{n-1} + (t-1)(n-1) \left(\left(\frac{\partial G_t}{\partial t} - dA \right) d\xi G_t^{n-2} - Ad\xi dG_t^{n-2} \right) \right).$$

Integration by parts cancels the first and the second terms in the integrand, so that

$$\begin{aligned} n(n-1) \int_0^1 dt (t-1) \operatorname{Str} (-dAd\xi G_t^{n-2} - Ad\xi dG_t^{n-2}) = \\ = n(n-1) \int_0^1 dt (1-t) d \operatorname{Str} (\xi d(G_t^{n-2} A)), \end{aligned}$$

and we arrived to the celebrated result for the non-Abelian anomaly [6, 7, 8, 11, 12, 13, 14]:

$$\omega_{2n-2}^1 = n(n-1) \int_0^1 dt (1-t) \operatorname{Str} (\xi d(AG_t^{n-2})). \quad (\text{C.1})$$

In $\mathcal{D} = 2n - 2$ dimensions non-Abelian anomaly is given by this $(2n - 2)$ -form, the higher dimensional analog of the eq. (5.2):

$$D * J_\xi^A \propto \omega_{2n-2}^1(\xi, A). \quad (\text{C.2})$$

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